Money Cycles*

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December 22, 2009

Abstract

This paper presents a general equilibrium model where money is essential and agents exchange in competitive markets. A fixed cost of production induces them to take vacations during some periods. Hence, money is saved to purchase consumption during vacations. We show that agents will choose to acquire and spend money in cycles of finite length, even though aggregates are stationary. At any given time, agents have different positions on the money cycle. Throughout the money cycle, agents decrease their consumption and decrease their sensitivity to the inflation tax. This explains why some sellers accept money even though everybody wants to escape the inflation tax – an old paradox. Despite this stark "hot potato" effect, inflation does not stimulate trade, and the Friedman rule is the optimal monetary policy.

Keywords:	money cycles, distribution of money holdings, hot potato effect of inflation
JEL Classification:	E41, D51

*We are very grateful to Costas Azariadis, Aleksander Berentsen, Danielle Catambay, Leo Kaas, Philipp Kircher, Andreas Kleiner, Ricardo Lagos, George Mailath, Iourii Manovskii, Guido Menzio, Cyril Monnet, Makoto Nakajima, Borghan Narajabad, Peter Norman, Stanislav Rabinovich, Liang Wang, Warren Weber, Randall Wright, and participants of the Fed Chicago Money Workshop for fruitful discussions. Carlo Strub thanks the Swiss National Science Foundation for financial support.

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1 Introduction

In standard microfounded models of divisible money, agents are identical at the end of every period. This makes it much easier to keep track of the distribution of money holdings which is either trivial (because imposed exogenously) or degenerate. One major reason why it is so hard to create a model with an endogenous non-trivial distribution of money holdings is the lack of double coincidents of wants, i.e. some agents want to be buyers, some want to be sellers today, what leads to complicated and untractable work-consumption patterns, but on the other hand is the nucleus of a proper microfoundation of money. The basic model of Lagos and Wright (2005) deals with this effect of random matching by assuming quasi-linear preferences in a centralized settlement market. Other papers use lotteries or indivisible labor markets to get the same linearities necessary to remove the wealth effect of money holdings. We deal with the non-convexities by supposing that trade can have predictable patterns, e.g. agents work Tuesday to Saturday and take Sunday to Monday off. This does not mean that all agents start their work pattern at the same date. On the contrary, some agents may want to take Friday–Saturday off, others may want to work from Wednesday to Sunday what yields to a non-trivial but tractable distribution of money holdings at each point in time.

We present a general equilibrium model where money is used as a medium of exchange and saving. We study stationary equilibria. All agents are homogeneous with identical preferences over the consumption of a single good which is produced at identical marginal and fixed cost. We assume that agents are anonymous and do not know trading histories of other agents. This excludes private credit and generates a demand for fiat money, an intrinsically useless medium of exchange. There is no uncertainty present in our model and lotteries cannot be enforced. We further assume that the production good is not storable.

For sufficiently high fixed cost of production, we show that it is optimal for agents to take vacations during some periods. To illustrate this result, imagine that workers need to commute for two hours every day from their home to work. Instead of working every day, they prefer to work five days a week and take a weekend off. Because of standard assumptions about the utility of consumption, agents not only want to consume during workdays but also during the weekends. Hence, they will work a little bit more every workday to save some money for the weekend (another reason why money is essential in this model). However, as these savings are in money only, they are subject to the inflation tax. We show that there exists a "hot potato" effect of inflation as agents reduce their consumption over time to save some money and thus reduce their exposure to inflation until the effect disappears, i.e. when they run out of money. On the other hand, agents increase their production over the money cycle – conditional on working – for the same reason they reduce consumption. Working later in a money cycle exposes the agents savings less to the inflation tax. It is important to note that although individual consumption and production may vary over the money cycle, aggregate consumption and production remain constant over time. This is possible because some agents work while others take their vacations.

We show that agents endogenously choose to acquire and spend money in *cycles of finite length* that have always a work period after starting with zero money holdings and a vacation period before spending all the available money. The reason is the inflation tax which constantly imposes costs on individuals as long as they hold money. Spending all available money after some time is better than saving forever because this brings total relief from this tax burden. Market clearing yields an equal measure of agents at each point of the support and hence produces a time invariant distribution of money holdings. This means that in each point in time, there are agents who are hurt severely by the inflation tax and want to get rid of their money, whereas others have the necessity to acquire it, e.g. to start a new money cycle. An explanation for an old paradox in monetary theory which asks why there are always some sellers willing to accept money even if it is a "hot potato."

Last but not least, we show that the Friedman rule is the optimal monetary policy.

Figure 1 presents a simple example of a money cycle. Agents work for one day and take the following two days off to avoid paying the fixed cost. They spend their savings from the workday on their vacation. Money holdings decrease monotonically over the cycle and agents spend all of their money during the final vacation day. We will show later, that this does not need to be the case in general and that much more complicated endogenous money cycles are supported by our model. The agents' consumption decreases throughout the cycle, as discussed above.

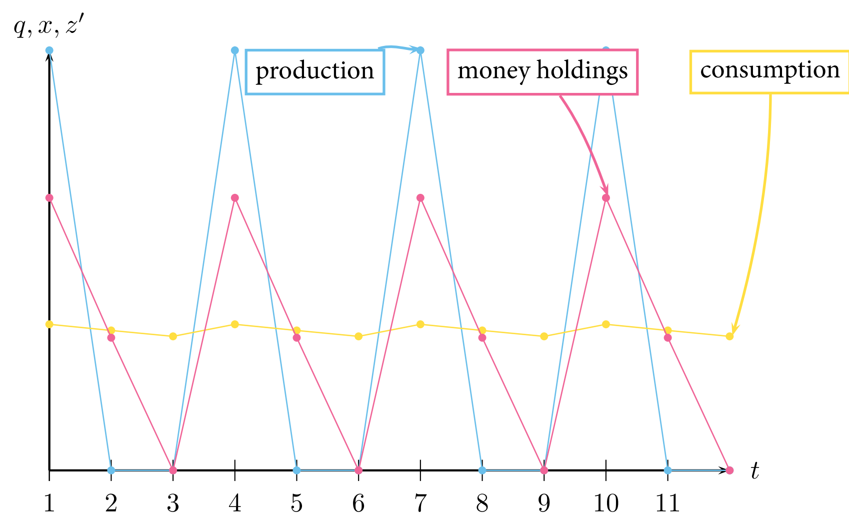

Figure 1: Simple money cycle

Literature As in the partial equilibrium models of Baumol (1952) and Tobin (1956), agents in our model have to deal with a trade-off between avoiding money acquisition costs (eg: transaction fees) and avoiding the cost of holding money (eg: inflation). In these types of models, it is inefficient for agents to acquire money (or consumption goods) every day. The fixed cost of production in our general equilibrium model leads to similar but non-trivial work patterns. With affine cost, our model replicates in a general equilibrium framework the pattern of acquiring money and spending all of it over several periods as in the standard Baumol-Tobin model. However, with increasing marginal costs, our model provides non-trivial work-plans and non-monotonic money cycles.

It is not totally surprising that the assumption of our technology (with fixed costs) leads to so called production bunching. Cooper and Haltiwanger (1993) study a model with fixed costs of production and inventory storage which leads to regular production cycles. Due to the fixed costs, an agent will bunch his production and live off the inventory until it runs down to zero. As a result, they show that consumption is positively correlated to production and appears to have the same cycling pattern. In our model without storable goods but money, however, this correlation does not exist as much more complicated plans are allowed.

A major reason for delaying production in our model is the inflation tax. Holding money is costly, because its value decreases over time unless monetary policy follows the Friedman rule. Hence, money is like a hot potato that nobody wants to hold too long. Similar to Berentsen, Camera, and Waller (2005), agents decrease consumption over each cycle because of the hot potato effect. However, in their paper, the cycles are exogenously given (they impose two "decentralized markets"), whereas our cycles arise endogenously. Several other search models of money including Lagos and Rocheteau (2005), Ennis (2009), or Nosal (2009) exhibit a hot potato effect by introducing various distortions to the terms of trade. Finally, Liu, Wang, and Wright (2009) generate a hot potato effect through search intensity whereas our paper has no search frictions.

Historically, it has proven rather difficult to study non-trivial distributions of money holdings with analytical tools. Indeed, the popular Lagos and Wright (2005) introduces a quasilinearity assumption in order to force a degenerate distribution of money holdings and obtain tractable results. This basic model has been expanded by quite a few people in order to get trivial distributions, e.g. distributions between different types of agents.[1] Molico (2006) and Dressler (2009) provide numerical results on distributions of money holdings in search and Arrow-Debreu models, respectively. Galenianos and Kircher (2008) construct a model in which agents are indifferent between holding several levels of money.

Menzio, Shi, and Sun (2009) is perhaps the closest paper to ours. However, the matching

[1] See for example Bhattacharya, Haslag, and Martin (2005), Boel and Camera (2006), or Berentsen and Strub (2009).

frictions in their model mean that agents do not proceed through deterministic cycles. In their model, agents choose to be buyers until their money holdings fall below a reservation level. After working, all agents hold the same amount of money. We construct examples in our model in which agents decision whether to work is non-monotonic in money holdings, and workers hold different amounts of money. Like their model, agents are unaffected by the distribution of money holdings in stationary equilibria.

Finally, our paper studies non-convexities in a novel manner that yields tractable results. Rogerson (1988) argued that labor is indivisible – firms prefer to hire one full-time worker rather than two half-time workers. To resolve the non-convexity, Rogerson allows agents to participate in work lotteries that pay a constant wage, but demand a stochastic amount of work. In our model, agents smooth out the non-convexity over time through work/vacation patterns. Previous work in monetary economics, such as Faig (2006, 2008) and Rocheteau, Rupert, Shell, and Wright (2008) has applied the Rogerson approach to resolve the non-convexities that arise when agents may choose to become either a buyer or a seller.

2 Environment

We construct a stationary general equilibrium model with infinite discrete time, where agents discount at rate β. Each period t, agents may produce any quantity $q_t \geq 0$ of the non-durable good at cost $c(q_t)$. They receive utility $u(x_t)$ from consuming x_t units of the good. If the agent produces more than he consumes, he sells the surplus for fiat money at the market price p_t. The agent may also consume more than he produces if he has enough money on hand to purchase the difference.

As it is a standard approach in the literature of monetary theory, we assume some form of anonymity to prevent agents from offering a personal IOU or private credit in exchange for output, promising to pay in the next work period. This form of anonymity makes a tangible medium of exchange essential. This medium of exchange is intrinsically useless, and we call it *money.*[2] To introduce monetary policy, let M_t be the aggregate stock of money in nominal terms at the beginning of period t. We assume that it evolves over time depending on a lump-sum transfer T_t issued by the government to all agents before starting to purchase goods.[3] We write $1 + \pi_{t+1} = M_{t+1}/M_t$.

If an agent holds m units of money at time t, and the continuation value of holding m

[2]For formal discussions on anonymity, we refer to Kocherlakota (1998), Wallace (2001), Corbae, Temzelides, and Wright (2003), or Aliprantis, Camera, and Puzzello (2006).

[3]The assumption about the lump-sum nature of these transfers is standard and important as we do not want to introduce an unnecessary exogenous distortion.

is $W_t(m)$, his optimal choices satisfy the Bellman equation

$$\begin{aligned} W_t(m) = & \max_{q\in\mathbb{R}_+, x\in\mathbb{R}_+, m'\in\mathbb{R}_+} u(x) - c(q) + \beta W_{t+1}(m') \\ & \text{s.t.} \quad (1+\pi_t)\frac{m'}{p_t} + x = q + \frac{m+T_t}{p_t}. \end{aligned} \tag{1}$$

We focus on stationary equilibria and a stationary monetary policy, so that the real transfer is stationary with $T = T_t/p_t$ and inflation is stationary with $\pi = \pi_t$. Since $M_t + T_t = (1+\pi) M_t$, or in real terms, $Z + T = (1+\pi) Z$, the real money stock can be expressed as $Z = T/\pi$. When we replace the nominal balances m_t with real balances $z_t = m_t/p_t$, the problem becomes stationary:

$$\begin{aligned} V(z) = & \max_{q\in\mathbb{R}_+, x\in\mathbb{R}_+, z'\in\mathbb{R}_+} u(x) - c(q) + \beta V(z') \\ & \text{s.t.} \quad (1+\pi) z' + x = z + q + T. \end{aligned} \tag{2}$$

We assume that the production cost $c(q)$ is strictly increasing, differentiable, and concave on $\mathbb{R}_+$, but allow for a discontinuity at $q = 0$ which represents a fixed cost. We assume that $c(0) = 0$ and that the marginal cost starts at 0, so that $\lim_{q\to 0^+} c'(q) = 0$. We also assume that u is strictly increasing, differentiable, and concave and satisfies the Inada conditions. In addition to the assumptions on u and c, we assume that when the fixed cost of c are subtracted off, the above dynamic programming problem has a solution. This ensures that the problems we study have solutions.

We write the optimal production quantity as $q(z)$. The consumption policy is $x(z)$. The distribution of real money holdings is F.

A *symmetric stationary equilibrium* in this environment is a tuple

$$[q(z), x(z), F(z), T]$$

such that

- the policies $q(z)$ and $x(z)$ solve the stationary problem above given T;
- goods and money markets clear so that supply equals demand:
$$\int q(z)\ dF(z) = \int x(z)\ dF(z) \quad \text{and} \quad \frac{T}{\pi} = \int z\, dF(z).$$
- the distribution of money holdings is stationary:
$$F(z') = \int I\{[z + q(z) - x(z) + T]/(1+\pi) \le z'\}\ dF(z)$$
which means that the measure of agents holding less than or equal to z' real balances of money at the beginning of the next period has to equal the measure that saved z' by working, consuming, and getting lump-sum transfers.

3 Non-Stationary Dynamic Programming

The stationary dynamic programming problem that agents face in our model is difficult to study because there is discrete choice, i.e. working versus taking a vacation, that creates a *non-convexity.* This means the value function may contain kinks, which makes it difficult to apply first-order conditions.

In the next section, we transform this (difficult) non-convex stationary problem into a (easy) convex non-stationary problem. But, before embarking on this, we develop a theory of non-stationary dynamic programming problems. Our main goal is to provide conditions under which the transformed value function is concave and differentiable, so that we can apply first-order conditions. To this end, we first generalize the contraction mapping theorem (also known as Banach fixed point theorem) to study sequences of (possibly different) contractions. Then, we write down a non-stationary optimization problem, and its equivalent recursive problem. Finally, we show that the recursive equations have a solution that is concave and differentiable.

To begin, we re-interpret the contraction mapping theorem. Recall that the standard theorem studies a single contraction $f : X \to X$. It finds that f has a unique fixed point which can be obtained by applying f infinitely many times to any point $x_0 \in X$. Instead of having one contraction, we study a sequence of contractions f_n. Rather than showing that there is a unique fixed point of these contractions (which does not make sense), we show that there is a unique sequence x_n such that $x_n = f_n(x_{n+1})$ for all n. The contraction mapping theorem can be viewed as a special case of our result when $f_n = f$. Conversely, our result is a special case of the contraction mapping theorem, where the metric space is the space of all bounded sequences in X, $\ell^\infty(X)$.

Theorem 1 (Infinitely Recursive Fixed Point Theorem) *Let f_n be a sequence of contractions of degree α on a complete metric space (X, d). There is a unique bounded sequence x_n in X such that $x_n = f_n(x_{n+1})$ for all n.* □

Proof Recall that $\ell^\infty(X)$ is the space of all bounded sequences in X, with the sup metric $d^\infty(x, y) = \sup_n d(x_n, y_n)$. Since X is complete, $\ell^\infty(X)$ is complete as well.

Let g be the self-map on $\ell^\infty(X)$ defined as follows

$$g(x) = f_0(x_1), f_1(x_2), \cdots, f_n(x_{n+1}), \cdots . \tag{3}$$

Notice that g is a contraction of degree[4] α, as

$$\begin{aligned} d^{\infty}\left[g\left(x\right),g\left(y\right)\right] &= \sup_{n}\, d\left[f_n\left(x_{n+1}\right),f_n\left(y_{n+1}\right)\right] \\ &\leq \sup_{n}\, \alpha\, d\left(x_{n+1},y_{n+1}\right) \\ &\leq \sup_{n}\, \alpha\, d\left(x_n,y_n\right) \\ &= \alpha\, \sup_{n}\, d\left(x_n,y_n\right) \\ &= \alpha\, d^{\infty}\left(x,y\right). \end{aligned}$$

Since g is a contraction on a complete metric space, it has a unique fixed point x by the contraction mapping theorem. Moreover, $x = g(x)$ if and only if $x_n = f_n(x_{n+1})$ for all n. We conclude that there is a unique bounded sequence x_n in X such that $x_n = f_n(x_{n+1})$ for all n. ■

We consider non-stationary problems in which the objective function and constraint set, which we collectively refer to as a *technology*, change over time. At time n, the agents solve the problem

$$\begin{aligned} v_n\left(x_n\right) = \sup_{\{x_t\}_{t=n+1}^{\infty}} \sum_{t=n}^{\infty} \beta^{t-n}\, u_t\left(x_t,x_{t+1}\right) \\ \text{such that} \quad x_{t+1} \in \Gamma_t\left(x_t\right), \end{aligned} \tag{4}$$

which is parameterized by $(X, \beta, \{u_t\}, \{\Gamma_t\})$. Each period has a different technology (u_t, Γ_t), where $u_t : \text{graph}(\Gamma_t) \to \mathbb{R}$ and Γ_t is a correspondence from X to X. We assume that $X \subseteq \mathbb{R}^n$ is convex and $\beta \in (0,1)$. There is no loss of generality in assuming that β is time-invariant; non-stationary discount rates can be accommodated by rescaling u_t.[5] We assume that Γ_t is monotonic so that for all $x \leq x'$ we have $\Gamma_t(x) \subseteq \Gamma_t(x')$ (we partially order X the standard way). The *Principle of Optimality*[6] is an informal idea that programs in this form can typically be rewritten in a recursive manner, using Bellman equations:

$$V_t\left(x\right) = \sup_{x' \in \Gamma_t(x)} u_t\left(x,x'\right) + \beta\, V_{t+1}\left(x'\right). \tag{5}$$

[4]Recall that f is a contraction of degree α if $d\left[f(x), f(y)\right] \leq \alpha\, d(x,y)$ where $\alpha \in (0,1)$.

[5] Rescaling u_t does mean that problems with $\beta > 1$ can be meaningfully studied; the Principle of Optimality has a boundedness condition that would break down in this case. The recursive problem would have a (nonsensical) solution.

[6]"Principle of Optimality – An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman, 1957, Ch. 3)

In economic applications of dynamic programming, Stokey and Lucas (1989) provide in Section 4.1 very general conditions for the equivalence of these problems that do not make any geometric assumptions on the primitives. They show that the value functions in the two problems are equivalent, and that the optimal policies are equivalent, i.e. the Principle of Optimality holds. While Stokey and Lucas write these optimization problems in a stationary manner, this is without loss of generality as time can be included as a state variable. It is only when properties such as convexity and differentiability are included – as in our case – that this interpretation is problematic.

Definition 1 A technology (u, Γ) is *smooth* if $u\,:\,\text{graph}\,(\Gamma)\,\rightarrow\,\mathbb{R}$ is bounded and continuous, $u\,(\cdot, x')$ is strictly increasing, strictly concave, and differentiable on the interior of its domain; and $\Gamma\,:\,X\,\rightarrow\,X$ is a non-empty valued, compact-valued, and continuous correspondence with a convex graph, with $\Gamma(x) \subseteq \Gamma(x')$ whenever $x \leq x'$. □

We call a non-stationary dynamic programming problem $(X, \beta, \{u_t\}, \{\Gamma_t\})$ *smooth* if the technology in each period (u_t, Γ_t) is smooth, and u_t is bounded uniformly.

Definition 2 Given a technology (u, Γ), a choice $x' \in \Gamma\,(x)$ is *feasible near* x if, for every $\hat{x}$ in some open neighborhood of x, that choice x' lies in $\Gamma\,(\hat{x})$. □

Theorem 2 *For every smooth problem, there exists a sequence of functions V_t that satisfies the Bellman equations in* (5) *and that is unique among the bounded sequences of continuous functions. Each function V_t is strictly increasing and strictly concave. Moreover, V_t is differentiable at $x \in$ int$\,(X)$ if an optimal choice at x is feasible near x.* □

Proof Define f_t as follows

$$[f_t\,(V)]\,(x) = \max_{x' \in \Gamma_t(x)} u_t\,(x) + \beta\,V\,(x')\,.$$

This definition coincides with the definition of the operator T in Stokey and Lucas (1989, p. 78). Their Theorem 4.6 implies that f_t is a well-defined self-map on bounded continuous functions, and that it is a contraction of degree β. Their proof of Theorem 4.8 (which we reproduce here) establishes that f_t is concavity-preserving. Let $x'\,(x)$ be an optimal policy, and write $x_\theta = \theta\,x_0 + (1-\theta)\;x_1$, and $x'_\theta = \theta\,x'\,(x_0) + (1-\theta)\;x'\,(x_1)$. Note that x'_θ is feasible at x_θ, by convexity of Γ_t. If V is concave then

$$\begin{aligned}
[f_t\,(V)]\,(x_\theta) &= u_t\,[x_\theta, x'\,(x_\theta)] + \beta\,V\,[x'\,(x_\theta)] \\
&\geq u_t\,(x_\theta, x'_\theta) + \beta\,V\,(x'_\theta) \\
&> \theta\;\{u_t\,[x_\theta, x'\,(x_0)] + \beta\,V\,[x'\,(x_0)]\} \\
&\quad + (1-\theta)\,\{u_t\,[x_\theta, x'\,(x_1)] + \beta\,V\,[x'\,(x_1)]\} \\
&= \theta\;[f_t\,(V)]\,(x_0) + (1-\theta)\;[f_t\,(V)]\,(x_1)\,,
\end{aligned}$$

which means that $f_t(V)(\cdot)$ is strictly concave. Moreover, $f_t(V)(\cdot)$ is strictly increasing, since Γ_t is monotonic and $u_t(\cdot, x'_t)$ is strictly increasing. (The agent can simply consume all of its excess capital, and is strictly better off.) To summarize, each contraction f_t preserves concavity and monotonicity. Moreover, they map such functions into strictly concave and strictly increasing functions.

Now we apply Theorem 1 on the sequence of contractions f_t of degree β, which are self maps on the complete metric space of continuous bounded functions. The theorem implies that there exists a sequence of continuous functions V_t such that $V_t = f_t(V_{t+1})$ for each t – that is, there is a solution to the system of Bellman equations. The theorem also asserts that this solution is unique among the bounded sequences of continuous functions.

Next, we apply Theorem 1 again on the subspace of increasing and concave functions. Since f_t is a sequence of contractions on this subspace, we conclude that V_t are increasing and concave also. Moreover, since $V_t = f_t(V_{t+1})$, and f_t transforms functions into strictly concave and strictly increasing functions, we conclude that V_t is strictly increasing and strictly concave.

Finally, we prove V_t is differentiable. This proof is based on Benveniste and Scheinkman (1979).[7] They apply a lemma, which is illustrated in Figure 2: if f is concave, g is differentiable, $f(x) \geq g(x)$ for all x, and $f(x_0) = g(x_0)$ then f is differentiable at x_0. Suppose x' is an optimal choice at time t given $x \in \text{int}(X)$. If x' is also feasible near x, then $V_t(x)$ is bounded below by $\hat{V}_t(x) = u_t(x, x') + \beta V_{t+1}(x')$. Notice that $\hat{V}_t$ is a differentable function because $u_t(\cdot, x')$ is differentiable and the last term is constant. Moreover, V_t is concave, so the lemma implies that V_t is differentiable at x. ■

4 Piecewise Smooth Dynamic Programming

The previous section developed a theory of dynamic programming that departed from the textbook approach by allowing the decision problem to be non-stationary. This section develops a theory of dynamic programming in stationary environments where the agent's objective is not differentiable. In many applications, kinks are introduced due to discrete choices, such as whether to work or take a vacation, to buy or sell, to allocate workers to one task or another. For simplicity of exposition, we focus on problems where the decision maker has a set of smooth technologies to choose from. We show that these problems can be transformed into non-stationary smooth problems, which can then be solved using the techniques described in the previous section.

[7] Like us, Benveniste and Scheinkman (1979) study non-stationary dynamic programming problems. Unlike us, they require $x' \in \text{int}[\Gamma_t(x)]$, whereas we only require x' to be feasible near x. Unlike us, they assume u_t is differentiable, but their proof only depends on $u_t(\cdot, x')$ being differentiable.

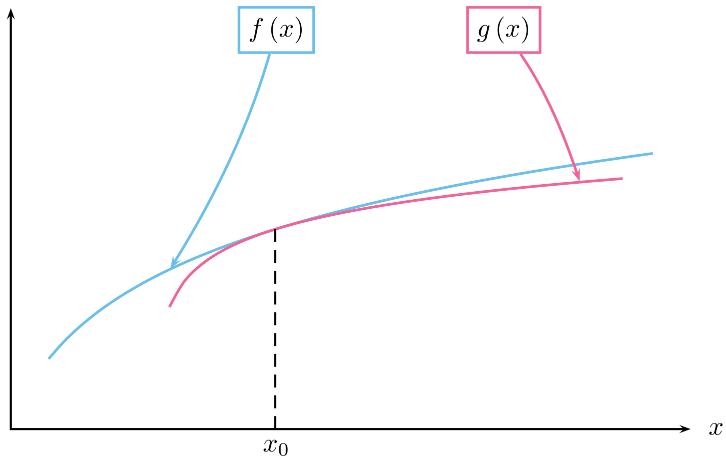

Figure 2: Illustration of the standard Lemma (Benveniste and Scheinkman, 1979)

There is a (possibly finite) set S of technologies, each with a utility function U^s and a correspondence giving a feasible set $\Gamma^s : X \to X$. The agent solves the problem

$$\bar{v}_n(x_n) = \sup_{\{s_t\}_{t=n}^{\infty}, \{x_t\}_{t=n+1}^{\infty}} \sum_{t=n}^{\infty} \beta^{t-n} U^{s_t}(x_t, x_{t+1})$$
$$\text{such that} \quad x_{t+1} \in \Gamma^{s_t}(x_t),$$

which can be written in recursive form as

$$V(x) = \sup_{s \in S} \sup_{x' \in \Gamma^s(x)} U^s(x, x') + \beta V(x').$$

As before, the standard Principle of Optimality results apply in this setting.

Definition 3 A stationary dynamic programming problem $(X, \beta, \{U^s\}, \{\Gamma^s\}, S)$ is *piecewise smooth* if each technology (U^s, Γ^s) is smooth and U^s is bounded uniformly. □

A piecewise smooth dynamic programming problem may thus be a combination of different smooth technologies.

We transform a stationary piecewise smooth problem $(X, \beta, \{U^s\}, \{\Gamma^s\}, S)$ into a family of non-stationary smooth problems $\left(X, \beta, \{\hat{u}_t\}, \left\{\hat{\Gamma}_t\right\}\right)$ parameterized by a sequence s_t in S by setting $\hat{u}_t = U^{s_t}$ and $\hat{\Gamma}_t = \Gamma^{s_t}$. We show that a solution to one of the transformed problems coincides, in a weak sense, with the solution to one of the piecewise smooth problems. This means that piecewise smooth problems can be solved with the tools of non-stationary problems.

Theorem 3 *Consider a piecewise smooth dynamic programming problem* $(X, \beta, \{U^s\}, \{\Gamma^s\}, S)$. *Fix* x_0*, and let* x_t^* *be an optimal sequence of state choices, and let* s_t^* *be an optimal sequence of discrete choices. Then*

(i) the transformed problem $\left(X, \beta, \{\hat{u}_t\}, \left\{\hat{\Gamma}_t\right\}\right)$ *for* s_t^* *satisfies the conditions of Theorem 2, and a unique bounded solution* $\hat{V}_t$ *exists which is increasing, concave, and differentiable.*

(ii) $V_t\left(x_t^*, x_{t+1}^*\right) = \hat{V}_t\left(x_t^*, x_{t+1}^*\right)$ *and* x_t^* *is an optimal solution to the transformed problem given* x_0. □

Proof (i) As all non-convexities in the problem are collected in the set S, fixing s_t^* returns a smooth problem as defined in Definition 1. Theorem 2 then applies for the transformed problem and proves that a unique bounded solution $\hat{V}_t$ exists which is increasing, concave, and differentiable.

(ii) *TODO: find a simple proof.* (Sketch of a long proof: we know that in the sequence problems, the discounted utility on the optimal path is unchanged when we add in the constraints. Then, applying the principle of optimality to the constrained and unconstrained problems shows that the solutions to the two sets of Bellman equations coincide on the optimal path.) ■

5 Stationary Equilibrium

In this section, we define money cycles, establish that in every stationary equilibrium, agents' decisions follow money cycles, and give qualitative results about consumption, production and money holdings throughout the cycle.

In the problem (2) defined above, agents have to decide whether to work. We allow for a fixed cost of production, which introduces a non-convexity at zero. Such dynamic programming problems are difficult to study because the standard results about the value function being differentiable and concave depend on convexity assumptions on the primitives. However, we can use the theory developed in the previous section to decompose the agent's problem into convex and discrete parts, which allows us to study the convex choices using first-order conditions.

We rewrite the problem in the language of the previous section. Each agent has two technologies: vacation ($s = 0$) and work ($s = 1$). As before, the state variable z is the agent's real balances. The vacation and work technologies are

$$\begin{aligned} U^0(z, z') &= u\left[z + T - z'(1+\pi)\right] \\ \Gamma^0(z) &= \left[0, (z+T)/(1+\pi)\right] \\ U^1(z, z') &= \max_{x \geq 0} u(x) - \bar{c}\left[x + z'(1+\pi) - z - T\right] \\ \Gamma^1(z) &= [0, \infty), \end{aligned}$$

where $\bar{c}$ is the same as c except the fixed cost is included at $q = 0$.

Let $\{s_t\}$ be an optimal sequence of work/vacation choices given some initial money holding z_0. Notice that $z' = 0$ is feasible for all z (that is, $z' \in \Gamma^s(z)$ for all (s, z)), so Theorem 2 implies that there is a sequence of strictly increasing, strictly concave and differentiable functions $\{\hat{V}_t\}$ that solve the Bellman equations for the non-stationary problem associated with $\{s_t\}$.

Since each value function $\hat{V}_t$ is differentiable, the Bellman equations satisfy the first-order conditions for optimality. On both work and vacation days, these are

$$(x): \qquad u'(x) = \lambda_b, \tag{6a}$$

$$(z'): \qquad (1+\pi)\,\lambda_b - \lambda_{z'} = \beta\,\hat{V}'_{t+1}(z'), \tag{6b}$$

where λ_b and $\lambda_{z'}$ are the Lagrange multiplier for the budget constraint and money holdings respectively. Since $z' \geq 0$ is a weak inequality constraint, it has the complementary slackness condition $\lambda_{z'}\, z' = 0$. On work days, there is an additional first-order condition

$$(q): \quad c'(q) = \lambda_b. \tag{7}$$

Note that it is not necessary to control for the non-negativity of the consumption and production quantities because of the Inada conditions.

Definition 4 We say that an agent's decisions $(\{q_t\}, \{x_t\}, \{z_t\})$ follow a *money cycle* of length n if n is the smallest number such that $z_t = z_{t+n}$ for all t. We say that the money cycle is *non-trivial* if $n > 1$. □

Theorem 4 *In every stationary monetary equilibrium away from the Friedman rule (i.e. for* $1 + \pi > \beta$*), agents' decisions follow a (possibly trivial) money cycle that contains* 0. □

PROOF Suppose $(\{q_t^*\}, \{x_t^*\}, \{z_t^*\})$ is an optimal solution to the agent's problem. Let $s_t^* = I(q_t^* > 0)$ be the work plan associated with this optimal solution, and consider the constrained value function $\hat{V}_t(z)$ associated with s_t^*. As discussed above, this constrained optimization problem is now convex, since the agent can no longer choose whether or not to pay the fixed cost of production. Therefore, $\hat{V}_t$ is strictly increasing, strictly concave, and continuously differentiable.

We show that z_t^* includes 0. By truncating the start of the sequences, we can repeat the argument to conclude that z_t^* includes 0 infinitely many times. Finally, in a stationary equilibrium, the decisions at z_t^* are identical, so the real balances must follow a finite cycle.

For the sake of contradiction, suppose that $z_t^* > 0$ for all t. In this case, each Lagrange multiplier $\lambda_{t,z'}^*$ is 0 by complementary slackness. Putting the resulting money first-order condition (6b) and the consumption first-order condition (6a) together give

$$(1+\pi)\; u'\left(x_t^*\right) = \beta\, \hat{V}_{t+1}'\left(z_{t+1}^*\right).$$

Applying the envelope theorem, the marginal value of money is

$$\hat{V}_{t+1}'\left(z_{t+1}^*\right) = u'\left(x_{t+1}^*\right)$$

Putting these together gives the Euler equation

$$u'\left(x_t^*\right) = \frac{\beta}{1+\pi}\, u'\left(x_{t+1}^*\right) = \left(\frac{\beta}{1+\pi}\right)^n u'\left(x_{t+n}^*\right). \tag{8}$$

When prices grow above the Friedman rule (i.e. $\beta < 1+\pi$), the first term on the right side converges to 0, so $u'\left(x_t^*\right) \to \infty$ and hence $x_t^* \to 0$. We now show this is suboptimal in a stationary monetary equilibrium. We provide separate proofs for the inflation and deflation cases.

Under inflation, $T > 0$ so that consuming $x_t = T$ every period is feasible (without relying on savings). When consumption drops below T, an agent would be strictly better of deviating to consuming T every period.

Under deflation, $T < 0$ but real money holdings z give a real return of $z\;(-\pi)/(1+\pi)$. When the real balance z is sufficiently large, the real return will be bigger than the lump sum tax $-T$, and the remainder, $x(z)$ can be consumed forever. Consider how much money is raised from working. Note that the worker has no last work date (since markets must clear). From (7) and (6a) we have

$$c'\left(q_t^*\right) = u'\left(x_t^*\right). \tag{9}$$

We already showed that the right side goes to ∞, which implies that q_t^* in work periods also goes to ∞. In these work periods, real money holdings z_t^* also go to ∞, and the agent can afford $x\left(z_t^*\right) \to \infty$ units of consumption forever. This means $x_t^* \to 0$ is suboptimal.

Finally, in the boundary case when $T = 0$ and $\pi = 0$, the Euler equation simplifies to

$$c'\left(q_{t+n}^*\right) = \beta^{-n}\, u'\left(x_t^*\right)$$

and the budget constraint simplifies to

$$z' + x = z + q.$$

Hence, the marginal cost increase over time as long as agents save money. However, the Inada condition $c'(q) \to \infty$ as $q \to \infty$ implies that there will be a time t where

$$c\left(q_t^*\right) > c\left(q_t^*/2\right) + c\left(q_t^*/2\right).$$

This cannot be an equilibrium as an agent is strictly better off producing less, which is inconsistent with the Euler equation we obtained from the assumption that the agent holds money. ■

Corollary 1 *No monetary equilibrium exist in trivial cycles.* □

Proof As shown above, every cycle includes zero. Together with stationarity this implies that all agents consume what they produce. Thus the money market clearing condition implies:

$$\begin{aligned} \frac{M_t}{P_t} &= \int z_t \, dF\left(m_t\right) \\ &= 0 \end{aligned}$$

It follows that $p_t = \infty$ and thus there exists no monetary equilibrium. ■

Corollary 2 *The stationary distribution F of (real balances of) money holdings has equal mass over a finite set.* □

Proof The support of the distribution of real balances coincides with the equilibrium sequence of real balances. Since each worker cycles through the sequence at the same pace, the measure at each point of the sequence is equal, so the stationary distribution is uniformly distributed over the support. ■

Let us finally examine the special case of the *Friedman rule,* i.e. the case where $(1 + \pi) = \beta$. In this case, agents are indifferent between holding money now or in the future. Further, from the Euler equation above, we know that consumption will be smooth over time, $x_t = x_1$. However, with the non-convexity in the cost function agents still need to concentrate their production efforts. Hence, they will produce produce $q_t \in \{0, \bar{q}\}$ and $\bar{q}$ is the quantity where the convex hull of the cost function meets the original function, i.e. where $\bar{q}\, c'\left(\bar{q}\right) = c\left(\bar{q}\right)$. Figure 3 displays the cost function.

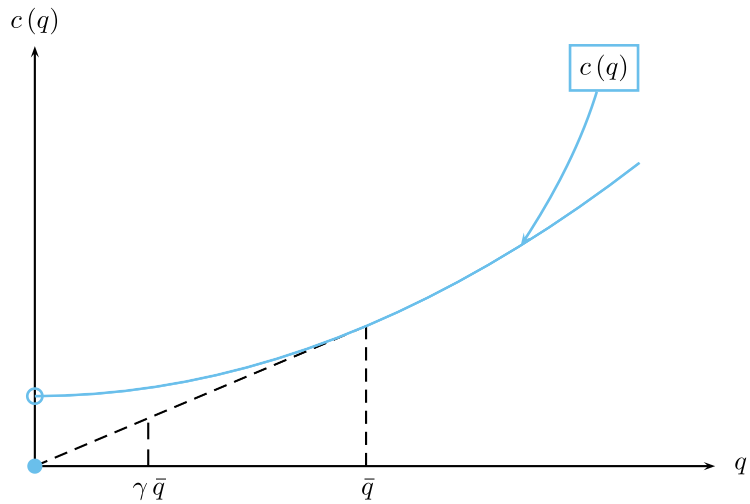

Figure 3: Cost function

If it is optimal for an agent to produce $\gamma\,\hat{q}$ on average (with $\gamma \in (0,1)$), he can only do so by alternating production and vacation periods. Due to our assumption of stationarity, a $\gamma \in \mathbb{Q}$ will lead to finite cycles. (For example, if $\gamma = a/b$, then the agent can work a days and then take $b-a$ days off.) If γ is an irrational number, there are no stationary equilibria at the Friedman rule, since the agent would like to choose a non-repeating sequence of work days.

Without loss of generality, we say that the start of the cycle is when agents hold no money.

Theorem 5 *In every stationary equilibrium, agents proceed through money cycles that*

(i) have decreasing consumption, with marginal utility increasing in proportion to the inflation tax, $\beta/(1+\pi)$. Hence at the Friedman rule, $\beta = 1+\pi$, agents have smooth consumption.

(ii) after eliminating vacations have production increasing throughout the money cycle, with (shadow) marginal cost increasing in proportion to the inflation tax.

(iii) begin with work and end with vacation. □

Proof (i) Previously, we found the Euler equation is satisfied whenever $z^*_{t+1} > 0$:

$$u'\left(x^*_t\right) = \frac{\beta}{1+\pi}\,u'\left(x^*_{t+1}\right) = \left(\frac{\beta}{1+\pi}\right)^n u'\left(x^*_{t+n}\right).$$

At the Friedman rule, $\beta = 1 + \pi$, the marginal utilities are equal over time, so the agent does perfect consumption smoothing. Away from the Friedman rule, $\beta < 1+\pi$ and marginal utilites are increasing.

(ii) When the agent works, marginal cost is related to marginal utility by the first-order condition,

$$c'\left(q_t^*\right) = u'\left(x_t^*\right).$$

Since the marginal utilities are increasing from the previous point, and the marginal cost increase in the same way (conditional on working).

(iii) The agent consumes the most in the first period. Since the agent works at some point in the money cycle (if anything is to be produced in the economy at all), the agent must need the payment from work to finance the first period's consumption.

If the agent worked in the last period, it would produce more than it produced in the first period. But since the agent spends all of its money in the last period, it would consume more in the last period than the first. But we already established that consumption decreases throughout money cycles. Therefore, the agent takes a vacation at the end of the money cycle. ∎

Figure 4 shows a non-trivial example of a money cycle including the distribution of money holdings.[8] For given functions (u and c) and parameters (β and π), we get the optimal plan $s^* = (1, 1, 0, 1, 0)$. Consumption x is decreasing over the cycle whereas the production quantity q is increasing, conditional on producing. Money holdings z' are non-zero (even at $t = 3$!) until the cycle ends with zero money holdings. Further, we see that consumption/production are not monotonic in money holdings; i.e. agents with more money do not work more, and do not consume more. The histogram of real balances presents a non-degenerate distribution of money holdings. Goods as well as money markets clear in each period of the cycle. Hence, aggregate consumption and production are stationary over time.

[8] The program code is available on the authors' websites.

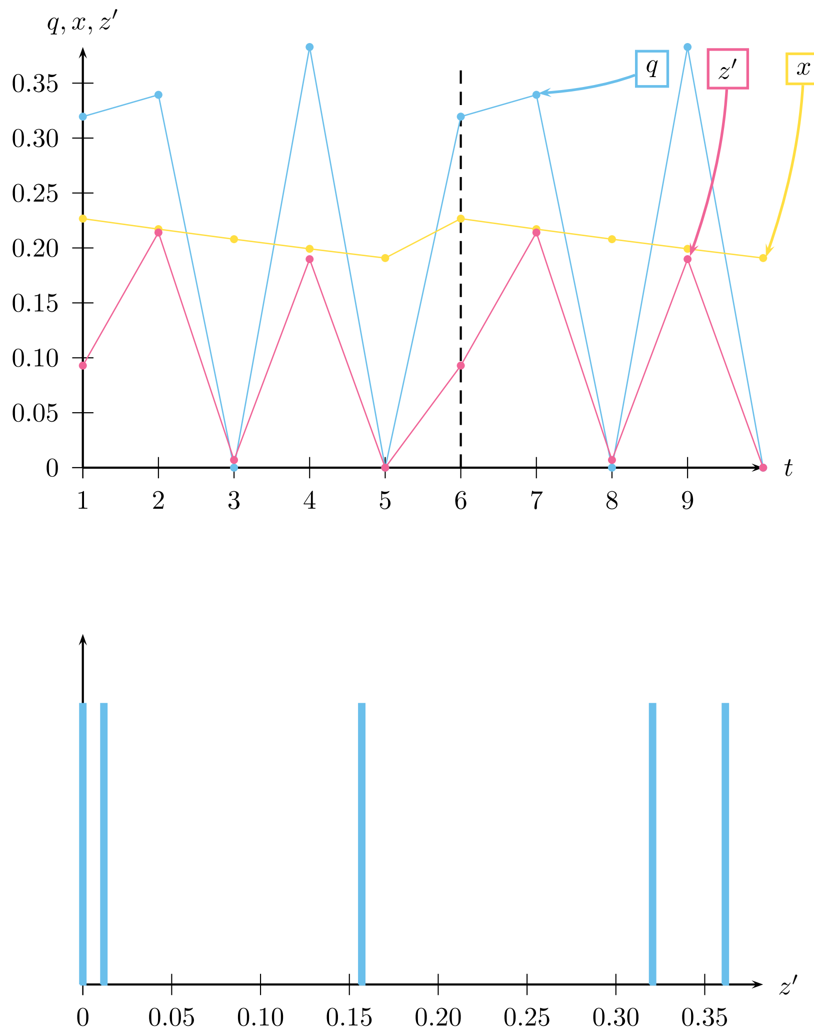

Figure 4: Equilibrium money cycle and related histogram of money holdings

6 Conclusion

This paper presents a general equilibrium model of money with homogeneous agents. A fixed cost of production introduces a non-convexity into the lifetime utility maximization problem. Despite the non-convexities in our model, we make the agents' decision problem tractable by decomposing it into convex and non-convex parts. Agents save money and take vacations to avoid the fixed cost. We show that in every stationary equilibrium, agents' money holdings proceed through finite cycles which can be non-trivial. This gives rise to non-trivial distributions of money holdings. We show that cycles begin with work and end with a vacation. Each agent's consumption decreases throughout the cycle due to the hot potato effect of inflation. Similarly production (conditional on working) increases throughout the cycle. The socially optimal monetary policy is the Friedman rule.

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A Sufficient Conditions for Equilibrium

In the body of the paper, we established that every stationary equilibrium exhibits money cycle behaviour. However, to know whether a candidate equilibrium is an equilibrium, we need to know whether there is any possible work plan that would give a higher payoff. Since the set of plans is infinite (indeed, uncountable), a brute force calculation is impossible. The following result establishes an upper bound on the length of optimal plans, so that only a finite set of plans needs to be checked.

Theorem 6 *Consider an equilibrium with inflation so that $T > 0$. If the agent holds real balances of z_1, then they will spend all of their money within the following number of periods,*

$$\left\lceil \log \frac{u'\left[\max\{z_1 + T, \psi^{-1}(0)\}\right]}{u'(T)} \Big/ \log \frac{\beta}{1+\pi} \right\rceil,$$

where $\psi(x) = u'(x) - c'(x - z_1 - T)$.

Firstly, this implies that the agent's value function is the upper envelope of a finite set of differentiable and concave functions (when the domain is restricted to the convex hull of the support of the real balances distribution F).

Secondly, since money cycles begin with $z_1 = 0$, the length of cycles is bounded by this expression at $z_1 = 0$. □

Proof First we show that $x_1 \leq \max\{z_1 + T, \psi^{-1}(0)\}$. If the agent does not work in the first period, then $x_1 \leq z_1 + T$. If the agent works, we will show that $x_1 \leq \psi^{-1}(0)$. Since the agent starts the cycle with z_1 real balances of money, the budget constraint implies that $q_1 \geq x_1 - z_1 - T$. The first-order conditions imply $u'(x_1) = c'(q_1)$. Moreover, since marginal cost is increasing, $c'(q_1) \geq c'(x_1 - z_1 - T)$. Thus, $u'(x_1) \leq c'(x_1 - z_1 - T)$, or equivalently, $\psi(x_1) \geq 0$. Since ψ is a decreasing function, $x_1 \leq \psi^{-1}(0)$.

Now suppose that $z_2, \ldots, z_n > 0$. We will put an upper bound on n for which this can be true. Under inflation, $x_n \geq T > 0$. By the Euler equation,

$$u'(x_1) = \left(\frac{\beta}{1+\pi}\right)^n u'(x_n).$$

Substituting the bounds for x_1 and x_n gives

$$u'\left[\max\{z_1 + T, \psi^{-1}(0)\}\right] \leq \left(\frac{\beta}{1+\pi}\right)^n u'(T),$$

which can be rearranged to the bound on n given above.