On the Friedman Rule with Heterogeneous Agents*

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Abstract

We consider a general equilibrium model where monetary policy has redistributive effects. Agents have stochastic preferences and face random buying and selling opportunities. In this environment, the Friedman Rule is just the second best policy. It requires to set the gross growth rate of the money supply equal to the discount factor of the most patient agents. Moreover, the Friedman Rule is not the only Pareto optimal monetary policy. We show that an elected central banker will not follow the Friedman Rule if the majority of the people is impatient.

Keywords: Friedman Rule, optimal monetary policy, majority voting

JEL Classification: E00, E50

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1 Introduction

This paper studies the optimal monetary policy in a model where changes in the money supply have redistributive effects. We generate these effects by assuming that agents have stochastic discount factors and face random buying and selling opportunities. At any point of time a proportion of all agents has high discount factors while the remaining agents have low ones. This framework generates heterogeneity between agents in all markets and across time. In equilibrium, patient agents hold more money than impatient ones. Consequently the inflation tax affects them differently.

The following results emerge from the model. In contrast to standard infinitely-lived-representative-agent models, the Friedman Rule, i.e. a policy that sets nominal interest rate to zero, cannot attain the first best allocation. Nevertheless, it is the second best policy, which requires to set the gross growth rate of the money supply equal to the discount factor of the most patient agents. Moreover, we illustrate that the Friedman Rule is not the only Pareto optimal policy.

Furthermore, we consider the real effects of a money supply shock. Unlike standard infinitely-lived-representative-agent models, under the Friedman Rule an unexpected increase in the money supply at the beginning of a period increases aggregate production.[1] The intuition behind this result is as follows. A sudden increase in the money supply increases all prices proportionally. This price increase is a proportional tax on the money holdings. Since impatient agents hold less cash, they are less taxed than patient agents. Consequently, after the lump sum injection has taken place, impatient agents are richer in real terms, what allows them to increase real consumption. On the other hand, patient agents are poorer in real terms after the injection. Nonetheless, they continue to consume the same real quantities because under the Friedman Rule they are not constrained by their money holdings. Consequently, under the Friedman Rule an unexpected increase in the money supply has a redistributive effect that increases aggregate output what means that the central bank faces a time inconsistency problem.

[1] An unexpected shock to the money supply has also no effects on aggregate output under the Friedman Rule in Berentsen et al. [2004].

On the contrary, a money supply shock has no real consequences if monetary policy deviates from the Friedman Rule. The intuition is that if the gross growth rate of the money supply is strictly larger than the discount factor of the most patient agents, all agents are constrained by their money holdings. In this case, after a positive supply shock, patient agents cannot keep their consumption constant because they have no extra cash to spend. Hence, they are forced to reduce real consumption. In fact, we show that they reduce their consumption exactly by the same amount than the impatient agents increase it, so that aggregate real consumption and production do not react to a money supply shock.

The presented model builds on the search-theoretic model of Lagos and Wright [forthcoming]. In contrast to their environment we assume that all markets are competitive as in Aruoba et al. [2003], Berentsen et al. [2004], or Rocheteau and Wright [forthcoming]. Our key innovation is that at any point of time a proportion of all agents has high discount factors while the remaining agents have low ones. The main consequence of this innovation is that there is a two-point distribution of money holdings in the market where the money supply shock takes place. Consequently, monetary policy has redistributive effects because the inflation tax affects agents differently. These effects are related to the ones in Berentsen et al. [2004], who also consider a model where agents hold different amounts of money in equilibrium. Berentsen et al. [2004] generate heterogeneity by assuming that agents have several random buying and selling opportunities in $j-1$ markets only before they can adjust their money holdings, whereas there is no heterogeneity between agents in the last market. Our framework guarantees heterogeneity in *all* markets.

There has been a recent interest in the redistributive effects of monetary policy. Among them are Haslag and Martin [2003] and Bhattacharya et al. [forthcomingb], who analyze the Friedman Rule in overlapping generations models. Concerning recent search-models of the Lagos-Wright type, Bhattacharya et al. [forthcominga] also considers a model with heterogeneous agents. The main differences to our model are the following. First, heterogeneity is generated by assuming that agents have different utility functions that do not change over time. In our model the discount factors follow a stationary first-order Markov process. Second, the terms of trade are determined by a generalized Nash solution in one market, whereas we use a Walrasian pricing.

In rest of the paper is organized as follows. In section 2, we present the environment. Section 3 analyzes monetary equilibria and the optimality of the Friedman Rule. We also include a numerical analysis of the inflation tax.

2 The Model

Time is discrete. There is a $[0, 1]$ continuum of infinite-lived agents. In each period two Walrasian markets open and close sequentially. Only one market, denoted by $j = 1, 2$, is open at any one time.

We assume there is one perishable good produced and consumed by all agents. Agents get utility $u(q_b)$ from consuming $q_b \geq 0$ of this good in the first market, where $u(0) = 0$, $u'(q_b) > 0$, $u'(0) = \infty$, $u'(+\infty) = 0$, $u''(q_b) < 0$, and $u'''(q_b) > 0$. In the last market each agent consumes and produces the same perishable good, getting utility $U(x)$ from x consumption, with $U(0) = 0$, $U'(x) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) < 0$. The cost of producing a good in both markets are linear, i.e. $c(q) = q$.

Whenever a market opens individuals receive a preference shock in discrete set $\mathbb{Z} \in \{s, b, sb\}$ that designates their willingness to consume and produce. This shock is summarized by the following transition probability matrix:

	s_2	b_2	sb_2
s_1	0	0	1
b_1	0	0	1
sb_1	$(1-\zeta)$	ζ	0

whereas s and b mean that agents want to sell or buy a good in the opening market, respectively. If agents get states s or b they only want to consume or to produce but not both. In other words, they either have infinite marginal cost of producing a good or zero marginal utility of consuming one. This happens only before the first market opens and an agent receives a consumption shock with probability ζ and a production shock with probability $1 - \zeta$. In the last market agents may want to consume and produce at the same time, sb, with certainty.

We assume our agents to be either patient or impatient. Every agent discounts future utility with a discount factor $\beta \in \left\{\beta^h, \beta^l\right\}$ where $1 > \beta^h > \beta^l > 0$. At the beginning of the second market each individual receives an idiosyncratic shock that determines her new discount factor. We assume that this shock follows a stationary finite state first-order Markov process, which is summarized in the following transition probability matrix[2]:

	β^h_{t+1}	β^l_{t+1}
β^h_t	π_{hh}	π_{hl}
β^l_t	π_{lh}	π_{ll}

where $\pi_{ab} \geq 0$ and $\pi_{ah} + \pi_{al} = 1$ for all $a \in \{l, h\}$. Furthermore, we assume that agents behave somewhat continous and thus are more likely to keep their degree of patience inherited from the previous period than to change it, i.e. $\pi_{hh}, \pi_{ll} \geq 1/2$.

Denote by n^l and n^h, with $n^l + n^h = 1$, the measures of agents with a low and high discount factor, respectively, in a steady state equilibrium. To derive these measures note that

$$n^l_{t+1} = (1 - \pi_{hh})\, n^h_t + \pi_{ll}\, n^l_t$$

The measure of agents with a low discount factor in period $t + 1$ is equal to the measure of agents with a high discount factor in period t who become impatient, $(1 - \pi_{hh})\, n^h_t$, plus the measure of agents with a low discount factor in period t who keep their low discount factor, $\pi_{ll}\, n^l_t$. In a steady state equilibrium $n^l_{t+1} = n^l_t = n^l$. Hence,

$$n^h = \frac{1 - \pi_{ll}}{1 - \pi_{hh} + 1 - \pi_{ll}} \qquad\qquad n^l = \frac{1 - \pi_{hh}}{1 - \pi_{hh} + 1 - \pi_{ll}} \tag{1}$$

While consumption and production goods are non-storable there is another object called *money* that can be transferred from one market to the other at no cost. Money is perfectly divisible and agents can hold any quantity $m \geq 0$.

[2]Note that this definition of the shock in the discount factors is not just made for the purpose of generalization. Instead, we will show later that the dependence on the old state—as defined in the Markov process—is essential for our results.

Money (e.g. a simple piece of paper) has no intrinsic value. Assuming anonymous trading people in the first market need money to trade consumption and production goods.[3] Because of the frictions arising from the buying and selling shocks buyers can buy goods and producers can sell them only if everyone accepts money. In the second market—the market where all agents can consume as well as produce at the same time—individuals could simply produce for themselves. But, they will not do so in general because they do not know their future states of $\mathbb{Z}$ and β. Hence, they need to adjust their money holdings depending their expected lifetime utility. We interpret this adjustment of the money holdings in the second market as a way to support self-insurance against the preference shocks coming in the next market. Note that due to the lack of credit in our model there exists only outside money but no inside money.

We assume that there is a monetary authority called *central bank* that prints fiat money for the economy at no cost. The central bank guarantees that each agent starts with an initial endowment of the fraction of the total nominal money stock $M/\left(n^h + n^l\right)$.[4] Further, we assume that the supply of money M changes according to $M_t = \mu\, M_{t-1}$. Imagine that money gets defective at the end of every period. In order to be able to use their money in the next period we assume that after trading in the second market all agents bring their money to the central bank's desk. There, the central bank reconditions the defective money. If she wants the money growth rate to increase, $\mu > 1$, the central bank reconditions an agent's money at a benefit $\tau = (\mu - 1)\, M_{t-1}$. A policy of zero money growth rate leads the central bank just to recondition the "old" money costlessly. In case the central bank wants to follow a policy of a decreasing money growth rate, $\mu < 1$, she reconditions each agent's money at a fixed cost $\tau = (\mu - 1)\, M_{t-1}$.[5]

If not otherwise mentioned we denote variables in our model as real variables. This makes it much easier to solve the model without loss of generality.

[3] See Wallace [2002] for a further discussion about anonymity in models of monetary exchange.

[4] We will see later that this initial endowment has no impact on our result in steady state.

[5] Lagos and Wright [forthcoming] have a different argumentation for a deflationary policy. They argue that the central bank has the power to introduce a lump sum tax τ. But, as Wallace [2002] pointed out, it is in reality difficult to collect the tax costlessly in an anonymous "underground" economy.

Figure 1 displays the timing of the decisions and shocks of the model.

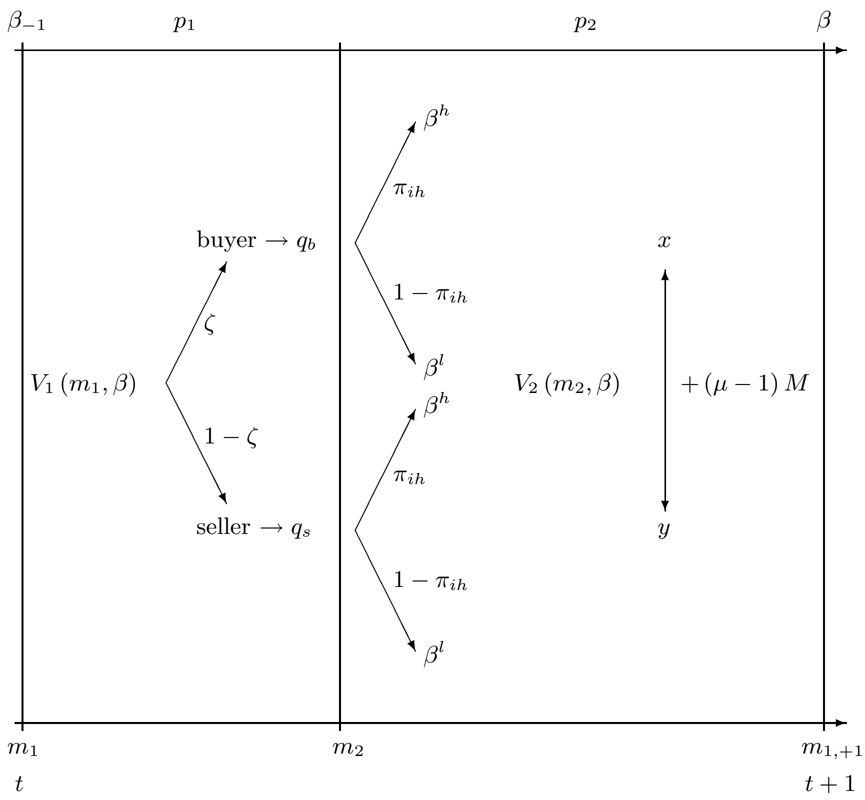

Figure 1: The Model ($i \in \{h, l\}$)

2.1 Consumption and Production

Let $p_{j,t}$ be the nominal price in market j in period t. We study equilibria where end-of-period real money balances are time-invariant:

$$\frac{M_t}{p_{2,t}} = \frac{M_{t-1}}{p_{2,t-1}} \tag{2}$$

We refer to it as a stationary equilibrium. This lets us omit the time subscript when understood, and study a representative period working backwards from last to first market—within one period.

2.1.1 The Last Market

Let $V_2(m_2, \beta)$ denote the *expected* value from trading in the last market of period t with m_2 real units of money and discount factor β. Define $V_1(m_{1,+1}, \beta)$ accordingly. Let x and y be the quantities bought respectively sold by an agent trading in the last market. In the last market agents choose how much to sell, y, how much to buy, x, and how much real money to take into the next period, $m_{1,+1}$. An agent with discount factor β and real money holdings m_2 at the beginning of the last market solves the following program:

$$V_2(m_2, \beta) = \max_{x, y, m_{1,+1}} U(x) - y + \beta V_1(m_{1,+1}, \beta)$$

$$\text{s.t.} \quad x + \frac{p_{1,+1}}{p_2} m_{1,+1} = y + m_2 + \frac{\tau}{p_2} \quad (3)$$

After substituting for y we get the following first-order condition for consumption:

$$U'(x) = 1 \quad (4)$$

According to (4), the marginal utility of buying equals the marginal cost of producing a good. Thus, trades are always efficient in the last market.

The first-order condition for the choice of real money holdings $m_{1,+1}$ satisfies

$$-\frac{p_{1,+1}}{p_2} + \beta V_1'(m_{1,+1}, \beta) = 0 \quad (5)$$

where $V_1'(m_{1,+1}, \beta) \equiv \frac{dV_1(m_{1,+1},\beta)}{dm_{1,+1}}$. Equation (5) implies that the amount of money agents take into the first market depends on their discount factor $\beta \in \{\beta^h, \beta^l\}$. Consequently, the distribution of beginning-of-period money holdings will have two mass points.

The envelope condition is

$$V_2'(m_2, \beta) = 1 \quad (6)$$

2.1.2 The First Market

Let q_b and q_s be the quantities bought by a buyer respectively sold by a seller trading in the first market. The expected lifetime utility of an agent with m_1

real money balances and the discount factor β at the opening of the first market is

$$V_1(m_1, \beta) = \zeta\, V_b + (1-\zeta)\, V_s \tag{7}$$

where

$$\begin{aligned} V_b &= u(q_b) + \pi_h\, V_2\left[(m_1 - q_b)\frac{p_1}{p_2}, \beta^h\right] + (1-\pi_h)\; V_2\left[(m_1 - q_b)\frac{p_1}{p_2}, \beta^l\right] \\ V_s &= -q_s + \pi_h\, V_2\left[(m_1 + q_s)\frac{p_1}{p_2}, \beta^h\right] + (1-\pi_h)\; V_2\left[(m_1 + q_s)\frac{p_1}{p_2}, \beta^l\right] \end{aligned}$$

and $\pi_h \in \{\pi_{hh}, \pi_{lh}\}$.

Sellers As a seller an agent chooses q_s to maximize V_s. Using (6) yields the following first-order condition

$$\frac{p_1}{p_2} \leq 1 \quad (= \text{if } q_s > 0) \tag{8}$$

We can interpret the left-hand side of (8) as the marginal revenue of a trade in the first market and the right-hand side as the marginal cost of producing a good.

A seller can acquire money in the first or in the second market and will do so at the lowest cost. Since sellers have the same linear production cost in both markets they are only indifferent if the prices in both markets are equal. In this case they are willing to supply all that is demanded, so the supply curve in the first market is flat. Using this we can state that the equilibrium price in the first market is

$$p_1 = p_2 \tag{9}$$

Buyers As a buyer, an agent chooses q_b to maximize V_b given her cash constraint

$$q_b \leq m_1 \tag{10}$$

Letting $\lambda \geq 0$ be the Lagrangian on the cash constraint and using (6) yields the following first-order conditions:

$$u'(q_b) - 1 - \lambda \leq 0 \quad (= 0 \text{ if } q_b > 0) \tag{11a}$$

$$\lambda (m_1 - q_b) = 0 \tag{11b}$$

Marginal Value of Money Finally, to derive the marginal value of money differentiate (7) with respect to m_1 and simplify to get

$$V_1'(m_1, \beta) = \zeta [u'(q_b) - 1] \frac{dq_b}{dm_1} + 1 \tag{12}$$

where we have used (6) and (9).

Finally, let q^* be the solution to $u'(q) = 1$ and let $m^* = q^*$. Then, we can state the following

Lemma 1. *In equilibrium, if*

(i) $m_1 \geq m^*$, *then* $\lambda^h = \lambda^l = 0$, $q_b^l = q_b^h = q^*$, *and* $V_1(m_1, \beta^i)$ *is linear.*

(ii) $m_1 < m^*$, *then* $\lambda^h, \lambda^l > 0$, $q_b^l, q_b^h < q^*$, *and* $V_1(m_1, \beta^i)$ *are concave.*

Lemma 1 shows that trades in the first market are efficient if and only if all buyers hold at least m^* units of money.

3 Equilibria

The key element of our model is that different discount factors generate a distribution of money holdings in the first market, i.e. every agent with a high discount factor enters the following period with m_1^h units of real money and those with a low one with m_1^l units. Then, agents receive preference shocks, which determine whether they are buyers or sellers in the first market. After trading all agents are randomly assigned a new discount factor at the beginning of the second market.

To determine the equilibria in our model we first state the following

Definition 1. A stationary monetary equilibrium is a time-invariant list $\{x, y, q_b^i, q_s^i\}$ and a sequence of prices and money holdings $\{p_1, p_2, \lambda^i, m_1^i\}$ for all $i \in \{h, l\}$ that satisfy (2)–(5), (7), (8) and (11).

We can now state our main proposition regarding the existence and uniqueness of monetary equilibria.

Proposition 1. *A stationary monetary equilibrium exists if and only if $\mu \geq \beta^h$. For $\mu = \beta^h$, we have $q_b^l < q_b^h = q^*$ and for $\mu > \beta^h$, $q_b^l < q_b^h < q^*$. If $\mu > \beta^h$, the equilibrium is unique.*

This proposition implies that there exist monetary steady state equilibria if and only if low-type agents are cash constrained. Patient agents are not constrained only if $\mu = \beta^h$, otherwise they are. The proposition also states that the money growth rate μ cannot be smaller than β^h. Analyzing the money holdings we can state that $m_1^h \geq m_1^l$ in any monetary equilibrium. The reason why high-type agents hold more money than low-type agents is that the opportunity cost of holding money are higher for low-type agents since they discount the future at a higher rate than patient agents.

3.1 Inflation Tax

In the following, we discuss how the inflation tax affects consumption and money holdings. Further, we study how welfare and expected lifetime utilities are affected by marginal changes in the money growth rate.

In a steady state monetary equilibrium first-period consumption satisfies

$$\frac{\mu - \beta^h}{\beta^h} = \zeta \left[u'\left(q_b^h\right) - 1 \right] \quad \text{and} \quad \frac{\mu - \beta^l}{\beta^l} = \zeta \left[u'\left(q_b^l\right) - 1 \right] \tag{13}$$

From this it is straightforward that inflation reduces first-period consumption and real money holdings of both patient and impatient agents, respectively:

$$\frac{\partial q_b^h}{\partial \mu} = \frac{1}{\beta^h \zeta u''\left(q_b^h\right)} < 0 \quad \text{and} \quad \frac{\partial q_b^l}{\partial \mu} = \frac{1}{\beta^l \zeta u''\left(q_b^l\right)} < 0 \tag{14}$$

It is less clear, however, how the inflation tax affects the variability of consumption and real money holdings. We measure the variability by the coefficient of variation. Figure 2 shows that the difference in the consumption quantities becomes smaller when inflation is growing. This is also confirmed by the coefficient of variation which is decreasing in the money growth rate μ in our numerical calculations.

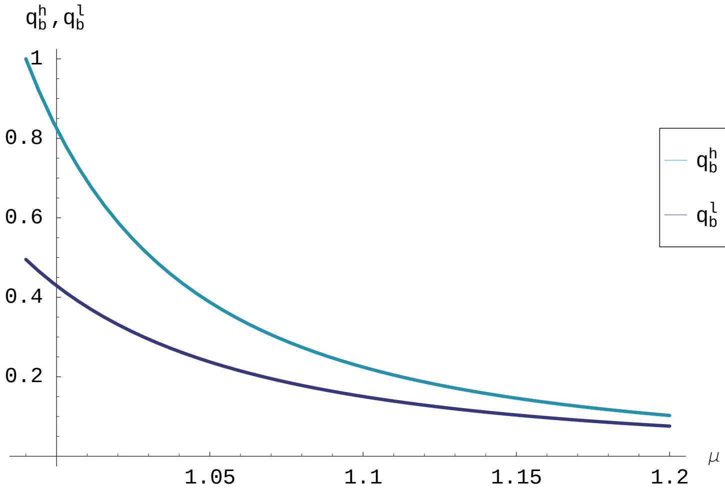

Figure 2: Effect of the Inflation tax on buying and selling quantities

In appendix A.3 we show that in equilibrium the expected discounted utilities of patient and impatient agents satisfy

$$V_1\left(m_1^h, \beta^h\right) = \frac{\left(1 - \beta^l\, \pi_{ll}\right) s_h + \beta^l\, \left(1 - \pi_{hh}\right)\, s_l}{1 - \beta^h\, \pi_{hh} - \beta^l\, \pi_{ll} - \beta^l\, \beta^h \left(1 - \pi_{ll} - \pi_{hh}\right)} \tag{15a}$$

$$V_1\left(m_1^l, \beta^l\right) = \frac{\left(1 - \beta^h\, \pi_{hh}\right) s_l + \beta^h\, \left(1 - \pi_{ll}\right)\, s_h}{1 - \beta^h\, \pi_{hh} - \beta^l\, \pi_{ll} - \beta^l\, \beta^h \left(1 - \pi_{ll} - \pi_{hh}\right)} \tag{15b}$$

where

$$s_h \equiv \Psi + \zeta\left[u\left(q_b^h\right) - q_b^h\right] + \frac{\tau}{p_2} - \pi_{hh}\, q_b^h\, (\mu - 1) - (1 - \pi_{hh})\, q_b^h\, \left(\mu\, \frac{q_b^l}{q_b^h} - 1\right)$$

$$s_l \equiv \Psi + \zeta\left[u\left(q_b^l\right) - q_b^l\right] + \frac{\tau}{p_2} - \pi_{ll}\, q_b^l\, (\mu - 1) - (1 - \pi_{ll})\, q_b^l\, \left(\mu\, \frac{q_b^h}{q_b^l} - 1\right)$$

$$\Psi \equiv U\left(x^*\right) - x^*$$

Welfare Using (15), the expected lifetime utilities then are

$$V_1\left(m_1^h, \beta^h\right) = \frac{\Psi + \zeta\left[u\left(q_b^h\right) - q_b^h\right] + (\mu - 1)\, n^l\left(q_b^l - q_b^h\right)}{1 - \beta^h} \tag{16a}$$

$$V_1\left(m_1^l, \beta^l\right) = \frac{\Psi + \zeta\left[u\left(q_b^l\right) - q_b^l\right] + (\mu - 1)\, n^h\left(q_b^h - q_b^l\right)}{1 - \beta^l} \tag{16b}$$

where we have used the fact that $\frac{\tau}{p_2} = (\mu - 1)\left(n^h\, q_b^h + n^l\, q_b^l\right)$.

From this we can show how the expected lifetime utilities change with the growth rate of the money supply μ.

$$\frac{dV_1\left(m_1^h, \beta^h\right)}{d\mu} = \frac{\zeta\left[u'\left(q_b^h\right) - 1\right]\frac{dq_b^h}{d\mu} + n^l\left(q_b^l - q_b^h\right) + (\mu - 1)\, n^l\, \frac{dq_b^h}{d\mu}\left(\frac{dq_b^l}{dq_b^h} - 1\right)}{1 - \beta^h} \tag{17a}$$

$$\frac{dV_1\left(m_1^l, \beta^l\right)}{d\mu} = \frac{\zeta\left[u'\left(q_b^l\right) - 1\right]\frac{dq_b^l}{d\mu} + n^h\left(q_b^h - q_b^l\right) + (\mu - 1)\, n^h\, \frac{dq_b^h}{d\mu}\left(1 - \frac{dq_b^l}{dq_b^h}\right)}{1 - \beta^l} \tag{17b}$$

The first term on the right-hand side of both equations is always negative because the marginal utilities are greater than 1 and $\frac{dq_b^i}{d\mu} < 0$. This term represents the loss of utility since consumption decreases as the inflation rate increases. The second term represents the wealth effect of the inflation tax. For high-type agents this effect is always negative since in equilibrium $q_b^l < q_b^h$ whereas it is always positive for low-type agents. In other words, an increase of the money growth rate has a negative impact on the expected lifetime utility patient agents because the inflation tax naturally affects rich people more than poor ones. Our numerical illustrations suggest that this effect is greater if there exist few rich agents because they possess relatively more of the total wealth than otherwise. The last term on the right-hand side represents the substitution effect of a change in the money growth rate. From (14) we have $\frac{dq_b^l}{dq_b^h} = \frac{\beta^h\, u''\left(q_b^h\right)}{\beta^l\, u''\left(q_b^l\right)} > 1$ since $u'''\left(q_b^i\right) > 0$. Thus, if $\mu > 1$, the substitution effect is negative for patient agents and they reduce their money holdings. In contrast, if $\mu < 1$ rich agents increase their money holdings. For poor agents the effect is inverse. In times of deflation impatient agents reduce their money holdings whereas they increase them in times of inflation. One explanation for this effect may be that in times of deflation rich individuals who are cash constrained can only hold their consumption level if they acquire more money.

We show the possible effects a change in the money growth rate has on the expected lifetime utility numerically in figure 3.

The fact, that inflation may have positive effects on the expected lifetime utility of impatient agents raises the natural question about the welfare maxi-

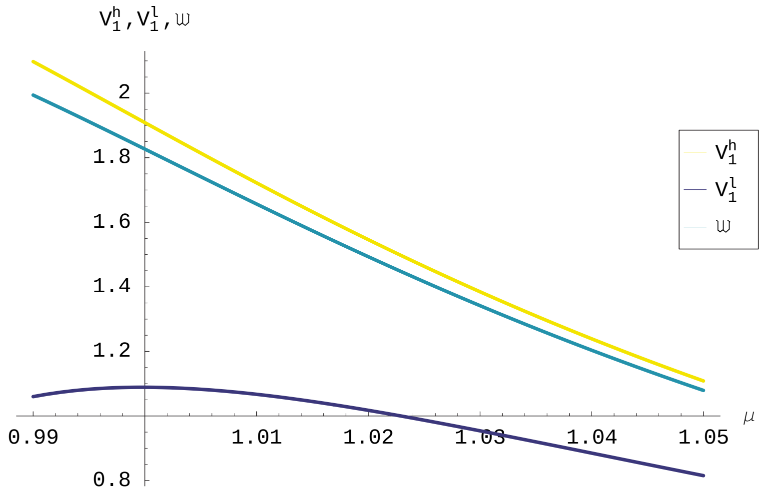

Figure 3: Inflation tax

mizing money growth rate. A weighted utilitarian welfare function in our model is

$$\mathcal{W} = n^h \left(1-\beta^h\right) V_1 \left(m_1^h, \beta^h\right) + n^l \left(1-\beta^l\right) V_1 \left(m_1^l, \beta^l\right)$$

Using (16) and simplifying yields

$$\mathcal{W} = \Psi + \zeta \left\{ n^h \left[u\left(q_b^h\right) - q_b^h \right] + n^l \left[u\left(q_b^h\right) - q_b^h \right] \right\}$$

Again, it is evident from consumption that $\frac{d\mathcal{W}}{d\mu} < 0$.

4 Optimal Monetary Policy

Friedman [1969] analyzed the effects of monetary injections into a hypothetical simple society consisting of a representative agent who discounts future consumption with a given discount factor. He found that the optimal policy is to set nominal interest rate to 0. In the standard representative agent model, from the Fisher equation, $1 + i = R(1 + \Pi)$, where R is the real return on a one-period bond and Π is the inflation rate, the Friedman Rule implies that the gross growth rate of the money supply is equal to the discount factor of the representative agent. Although we have two types of agents with different expected discount factors and no market for real bonds, the same logic applies. The real returns in the first market are $R^h = \frac{1}{\beta^h}$ and $R^l = \frac{1}{\beta^l}$ where $R^h < R^l$. Since in a competitive market for real one-period bonds the lower price will prevail, we will refer to a policy where $\mu \rightarrow \frac{1}{R^h}$ as the Friedman Rule.

Evidently, the Friedman Rule implements only the second best monetary policy in our model because impatient agents will never be able to consume the efficient quantity. However, in contrast to a representative agent model, the Friedman Rule is not the only Pareto efficient policy. In fact, if we denote by $\mu^l > \beta^h$ the monetary policy that maximizes the lifetime utility of the impatient agents, any growth rate $\mu \in \left[\beta^h, \mu^l\right]$ is Pareto efficient. This raises interesting monetary policy issues. Suppose that agents would elect at the end of each period a central banker. If $n^l > n^h$ then the elected central banker will not follow the Friedman Rule but he will instead follow her voters' optimal monetary policy, which is μ^l. Also a Rawlsian social planner will choose μ^l to maximize the expected lifetime utility of the poor...

5 Conclusion

This paper has presented a framework that combines stochastic discount factors with a search-theoretic model of monetary exchange. Our key innovation is that agents are heterogeneous in all subperiods, without losing the last market's property of producing degenerate money holdings. This allows us to examine the optimal monetary policy when money holdings differ within a period but also across periods.

The following results emerge from the model. First, the Friedman Rule—setting the money growth rate equal to the expected discount factor of the most patient agents—is the second-best policy. Furthermore, it is not the only Pareto optimal policy. Second, unexpected shocks to the money supply are non-neutral. In contrast to standard infinitely-lived-representative-agent models, they are also non-neutral under the Friedman Rule. Third, a positive shock to the money supply increases aggregate output and welfare if inflation is not too high. Again, in contrast to all models that we are aware of, a positive shock increases aggregate output under the Friedman Rule too.

A Appendix

A.1 Proof of Lemma 1

Constraints are not binding If $\lambda^h = \lambda^l = 0$ then from (9) and (11) we have

$$u'\left(q_b^h\right) = u'\left(q_b^l\right) = 1 \quad \Rightarrow q_b^h = q_b^l = q^*$$

The amount of money agents spend are $q^* = m^*$.

To examine linearity of V_1 note that from (12) we have

$$V_1'\left(m_1, \beta^h\right) = V_1'\left(m_1, \beta^l\right) = 1 \tag{18a}$$

Constraints are binding If $\lambda^h > 0$ and $\lambda^l > 0$. then from equations (11) we have

$$u'\left(q_b^i\right) \leq 1 + \lambda^i$$
$$m_1^i = q_b^i$$

for all $i = \{h, l\}$. Hence, trades are always inefficient.

To examine the shape of V_1 note that from (12) we have

$$V_1'\left(m_1, \beta^i\right) = \zeta\left[u'\left(q_b\right) - 1\right] + 1 \tag{19a}$$

Since $\frac{dq_b}{dm_1} = 1$. Note that $V_1\left(m_1, \beta^i\right)$ is concave for all $m_1 < m^*$ since $u''\left(q_b\right) < 0$. □

A.2 Proof of Proposition 1

The Lagrangians of both types of agents λ^h and λ^l are of central importance. There are four cases.

Case 1: $\lambda^h = \lambda^l = 0$ In this case $q_b^h = q_b^l = q^*$. From (12) we have

$$V_1'\left(m_1, \beta^i\right) = 1 \quad i = l, h$$

These conditions state that if agents take a unit of money into the first market but do not intend to spend it in this market (because they still have enough money), then the value of this extra unit of money is the goods it buys in the last market. Substituting these equations into the first-order conditions for the choice of money holdings (5) and backdating yields

$$-\frac{p_2}{p_{2,-1}} + \beta^i = 0 \quad i = l, h$$

In a steady state equilibrium, $\frac{M}{p_2} = \frac{M_{-1}}{p_{2,-1}}$, implying $\frac{p_2}{p_{2,-1}} = \mu$. Using this fact yields

$$-\mu + \beta^i = 0 \quad i = l, h \tag{20}$$

Note that in a monetary equilibrium (20) can hold only if $\mu = \beta^h = \beta^l$, which is a contradiction because $\beta^h > \beta^l$. Consequently, there exists no monetary equilibrium where $\lambda^h = \lambda^l = 0$.

Case 2: $\lambda^h > \lambda^l = 0$ In this case $q_b^h < q_b^l = q^*$. From (12) it follows that

$$\begin{aligned} V_1'\left(m_1, \beta^h\right) &= \zeta\left[u'\left(q_b^h\right) - 1\right] + 1 \\ V_1'\left(m_1, \beta^l\right) &= 1 \end{aligned}$$

Substituting these equations into the first-order conditions for the choice of money holdings (5) and backdating yields

$$\begin{aligned} -\frac{p_2}{p_{2,-1}} + \beta^h\left[\zeta\left[u'\left(q_b^h\right) - 1\right] + 1\right] &= 0 \\ -\frac{p_2}{p_{2,-1}} + \beta^l &= 0 \end{aligned}$$

In a steady state equilibrium the conditions for a monetary equilibrium therefore are

$$\frac{\mu - \beta^h}{\beta^h} = \zeta\left[u'\left(q_b^h\right) - 1\right] \tag{21a}$$

$$\mu = \beta^l \tag{21b}$$

Note that because of (21a) $\mu < \beta^h$. This immediately implies that $q_b^h > q^*$ which is a contradiction. Hence a monetary equilibrium with $\lambda^h > \lambda^l = 0$ cannot exist.

Case 3: $\lambda^l > \lambda^h = 0$ In this case $q^l_{1b} < q^h_{1b} = q^*$. From (12) we have

$$V_1'\left(m_1, \beta^h\right) = 1$$
$$V_1'\left(m_1, \beta^l\right) = \zeta\left[u'\left(q^l_b\right) - 1\right] + 1$$

Substituting these equations into the first-order conditions for the choice of money holdings (5) and backdating yields

$$-\frac{p_2}{p_{2,-1}} + \beta^h = 0$$
$$-\frac{p_2}{p_{2,-1}} + \beta^l\left[\zeta\left[u'\left(q^l_b\right) - 1\right] + 1\right] = 0$$

In a steady state equilibrium $\frac{1}{p_{2,-1}} = \frac{\mu}{p_2}$ and $\frac{1}{p_{2,-1}} > 0$ implying

$$\beta^h - \mu = 0$$
$$\frac{\mu - \beta^l}{\beta^l} = \zeta\left[u'\left(q^l_b\right) - 1\right]$$

For $\beta^h = \mu$, there exist an infinity of monetary equilibria, one for each value of p_2. Thus, a monetary steady state equilibrium with $\lambda^l > \lambda^h = 0$ and $m^h_1 > 0$ exists only if the above conditions hold.

Because of strict concavity of $u(q)$ there is a unique value q^l_b that solves the equilibrium condition above. Since $q^h_b = q^*$, the equilibrium price is determined as follows

$$n^l\, q^l_b + n^h\, q^* = \frac{M}{p_1}$$

Then the budget constraints yield m^l_1 and m^h_1, as follows

$$m^l_1 = q^l_b \quad \text{and} \quad m^h_1 = q^h_b = q^*$$

Finally, p_2 and λ^h are given by (9) and (11), respectively.

Case 4: $\lambda^l > \lambda^h > 0$ From (12) it follows that,

$$V_1'\left(m_1,\beta^h\right)=\zeta\left[u'\left(q_b^h\right)-1\right]+1$$
$$V_1'\left(m_1,\beta^l\right)=\zeta\left[u'\left(q_b^l\right)-1\right]+1$$

Substituting these equations into the first-order conditions for the choice of money holdings (5) and backdating yields

$$-\frac{p_2}{p_{2,-1}}+\beta^h\left[\zeta\left[u'\left(q_b^h\right)-1\right]+1\right]=0$$
$$-\frac{p_2}{p_{2,-1}}+\beta^l\left[\zeta\left[u'\left(q_b^l\right)-1\right]+1\right]=0$$

In a steady state equilibrium the conditions for a monetary equilibrium therefore are

$$\frac{\mu-\beta^h}{\beta^h}=\zeta\left[u'\left(q_b^h\right)-1\right] \tag{22a}$$
$$\frac{\mu-\beta^l}{\beta^l}=\zeta\left[u'\left(q_b^l\right)-1\right] \tag{22b}$$

Equations (22) yield unique values of $q_b^i \in \{h,l\}$. The equilibrium money holdings m^i and the price level p_1 can be derived by solving the following three equations

$$q_b^i=m_1^i \quad i=l,h$$
$$n^h\,m^h+n^l\,m^l=\frac{M}{p_1}$$

Then, $p_1=p_2$. The quantities in the second market satisfy...

Thus, if $\mu>\beta^h\,b^h$ a unique monetary equilibrium exists. □

A.3 Expected Lifetime Utilities

From (3) we derive the maximized expected value function for the second market:

$$V_2\left(m_1,\beta^i\right)=U\left(x^*\right)-\left[x^*+\mu\,m_{1,+1}^i-(\mu-1)\,\frac{M}{p_2}-m_2^i\right]+\beta^i\,V_1\left(m_{1,+1},\beta^i\right)$$

Inserting this equation into (7) yields

$$
\begin{aligned}
&V_1\left(m_1^h,\beta^h\right)= \\
&\zeta\Bigg\{u\left(q_b^h\right)+\pi_{hh}\Big[\Psi-\mu\, m_{1,+1}^h+(\mu-1)\,\frac{M}{p_2}+m_1^h-q_b^h+\beta^h\,V_1\left(m_{1,+1}^h,\beta^h\right)\Big] \\
&\quad+(1-\pi_{hh})\left[\Psi-\mu\, m_{1,+1}^l+(\mu-1)\,\frac{M}{p_2}+m_1^h-q_b^h+\beta^l\,V_1\left(m_{1,+1}^l,\beta^l\right)\right]\Bigg\} \\
&(1-\zeta)\Bigg\{-q_s^h+\pi_{hh}\Big[\Psi-\mu\, m_{1,+1}^h+(\mu-1)\,\frac{M}{p_2}+m_1^h+q_s^h+\beta^h\,V_1\left(m_{1,+1}^h,\beta^h\right)\Big] \\
&\quad+(1-\pi_{hh})\left[\Psi-\mu\, m_{1,+1}^l+(\mu-1)\,\frac{M}{p_2}+m_1^h+q_s^h+\beta^l\,V_1\left(m_{1,+1}^l,\beta^l\right)\right]\Bigg\}
\end{aligned}
$$

where $\Psi\equiv U\left(x^*\right)-x^*$. Note that we solve only for the high-type agents here. For the low-type agents the results are similar. Simplifying the above equation, we can write for both types of agents

$$
V_1\left(m_1^h,\beta^h\right)=s^h+\pi_{hh}\,\beta^h\,V_1\left(m_{1,+1}^h,\beta^h\right)+(1-\pi_{hh})\,\beta^l\,V_1\left(m_{1,+1}^l,\beta^l\right) \tag{23a}
$$

$$
V_1\left(m_1^l,\beta^l\right)=s^l+\pi_{lh}\,\beta^h\,V_1\left(m_{1,+1}^h,\beta^h\right)+(1-\pi_{lh})\,\beta^l\,V_1\left(m_{1,+1}^l,\beta^l\right) \tag{23b}
$$

where

$$
\begin{aligned}
s_h &\equiv \Psi+\zeta\left[u\left(q_b^h\right)-q_b^h\right]+\frac{\tau}{p_2}-\pi_{hh}\,q_b^h\,(\mu-1)-(1-\pi_{hh})\,q_b^h\left(\mu\,\frac{q_b^l}{q_b^h}-1\right) \\
s_l &\equiv \Psi+\zeta\left[u\left(q_b^l\right)-q_b^l\right]+\frac{\tau}{p_2}-\pi_{lh}\,q_b^l\left(\mu\,\frac{q_b^h}{q_b^l}-1\right)-(1-\pi_{lh})\,q_b^l\,(\mu-1)
\end{aligned}
$$

In a stationary equilibrium $m_{1,+1}^i=m_1^i$. Furthermore, all agents are cash constrained, i.e. $m_1^i=q_b^i$, $i\in\{h,l\}$. Solving for the value functions of the two types of agents we get (15).

Inflation Tax To find out how the money growth rate marginally affects the expected lifetime utilities we take the first derivative of (15) to have

$$\frac{\partial V_1\left(m_1^h, \beta^h\right)}{\partial \mu} = \Bigg(\left(1-\beta^l\right)(1-\pi_{hh}) \left\{(1-\pi_{hh}-\pi_{ll})\left[q_b^h - q_b^l + \mu\left(\frac{\partial q_b^h}{\partial \mu} - \frac{\partial q_b^l}{\partial \mu}\right)\right] + \left(\frac{\partial q_b^h}{\partial \mu} - \frac{\partial q_b^l}{\partial \mu}\right)\right\} + \zeta\,(2-\pi_{hh}-\pi_{ll}) \left\{\left(1-\pi_{ll}\,\beta^l\right)\frac{\partial q_b^h}{\partial \mu}\left[u'\left(q_b^h\right)-1\right] - \beta^l(1-\pi_{hh})\frac{\partial q_b^l}{\partial \mu}\left[u'\left(q_b^l\right)-1\right]\right\}\Bigg) \Bigg/ \Bigg((2-\pi_{hh}-\pi_{ll})\left[1-\beta^h\,\pi_{hh} - \beta^l\,\pi_{ll} - \beta^h\,\beta^l\,(1-\pi_{hh}-\pi_{ll})\right]\Bigg) \quad (24)$$

and

$$\frac{\partial V_1\left(m_1^l, \beta^l\right)}{\partial \mu} = \Bigg(\left(1-\beta^h\right)(1-\pi_{ll}) \left\{(1-\pi_{hh}-\pi_{ll})\left[q_b^l - q_b^h + \mu\left(\frac{\partial q_b^l}{\partial \mu} - \frac{\partial q_b^h}{\partial \mu}\right)\right] + \left(\frac{\partial q_b^l}{\partial \mu} - \frac{\partial q_b^h}{\partial \mu}\right)\right\} + \zeta\,(2-\pi_{hh}-\pi_{ll}) \left\{\left(1-\pi_{hh}\,\beta^h\right)\frac{\partial q_b^l}{\partial \mu}\left[u'\left(q_b^l\right)-1\right] - \beta^h(1-\pi_{ll})\frac{\partial q_b^h}{\partial \mu}\left[u'\left(q_b^h\right)-1\right]\right\}\Bigg) \Bigg/ \Bigg((2-\pi_{hh}-\pi_{ll})\left[1-\beta^h\,\pi_{hh} - \beta^l\,\pi_{ll} - \beta^h\,\beta^l\,(1-\pi_{hh}-\pi_{ll})\right]\Bigg) \quad (25)$$

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